Binomial distribution

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GE02: day 3 part 1

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Problem

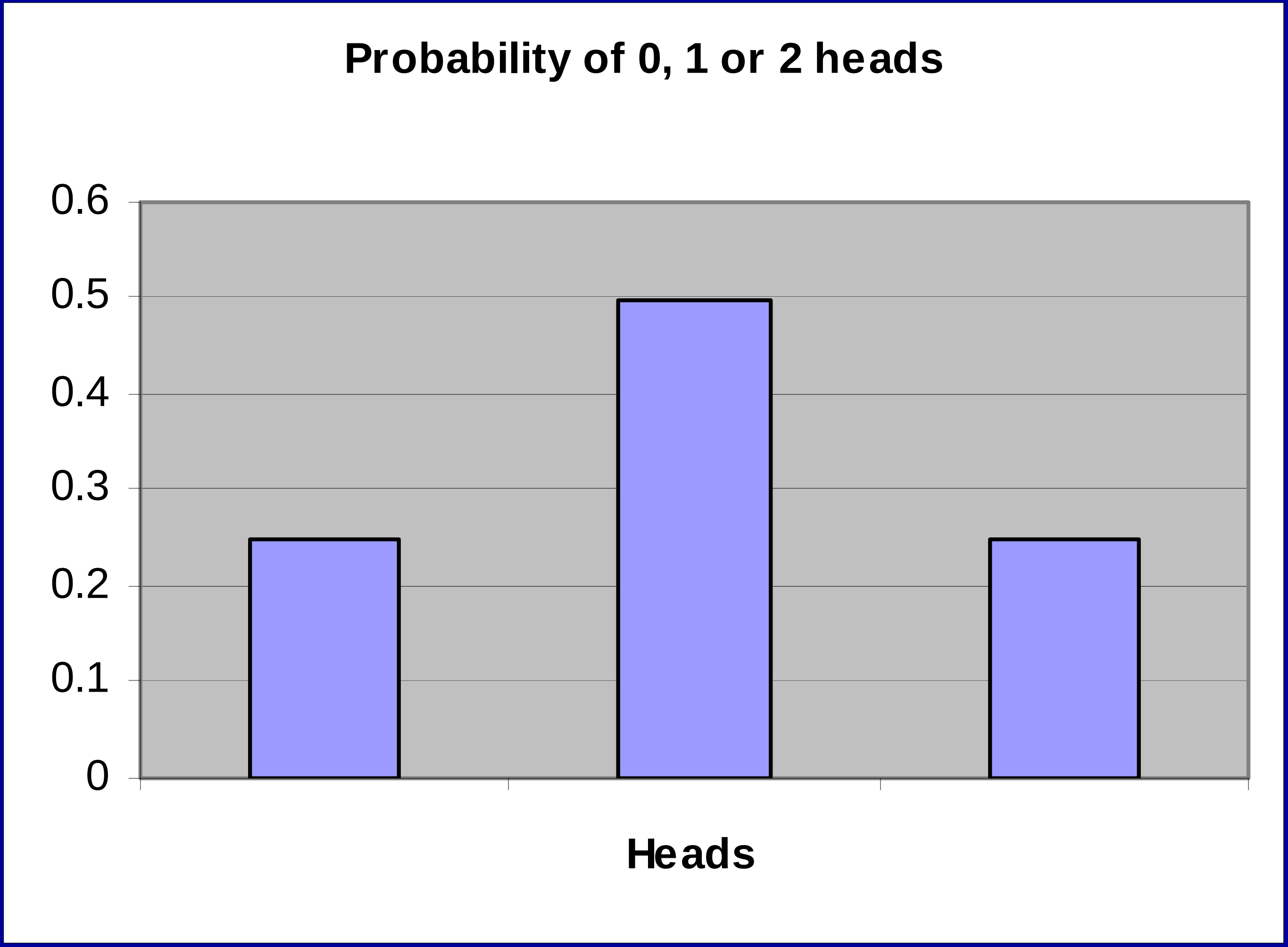
- Consider an experiment in which a coin is tossed 2 times. What is probability to have
  - No heads
  - 1 head
  - 2 heads

Solution

- Probability of any particular outcome is
  - $2^{-n} = ½\ ½ \ldots ½$ (n times)
  - Four outcomes – **HH**, **HT**, **TH** and **TT**, each having probability ¼

Event	Outcomes	Probability
No heads	{**TT**}	¼
1 head	{**HT**, **TH** }	¼ + ¼ = ½
2 heads	{**HH**}	¼
Total		4 ¼ = 1.0

Probability distribution



Problem

- Consider an experiment in which a coin is tossed 4 times. What is probability to have
  - No heads
  - 1 head
  - 2 heads
  - 3 heads
  - 4 heads

Idea of solution

- $2^n = 2^4 = 16$ experimental outcomes possible
- Every outcome has probability $(½)^4$
- The probability to have *k* "heads" is
  - $(½)^4$ x [# of outcomes leading to *k* "heads"]
- Write down all 16 outcomes and count heads

16 outcomes

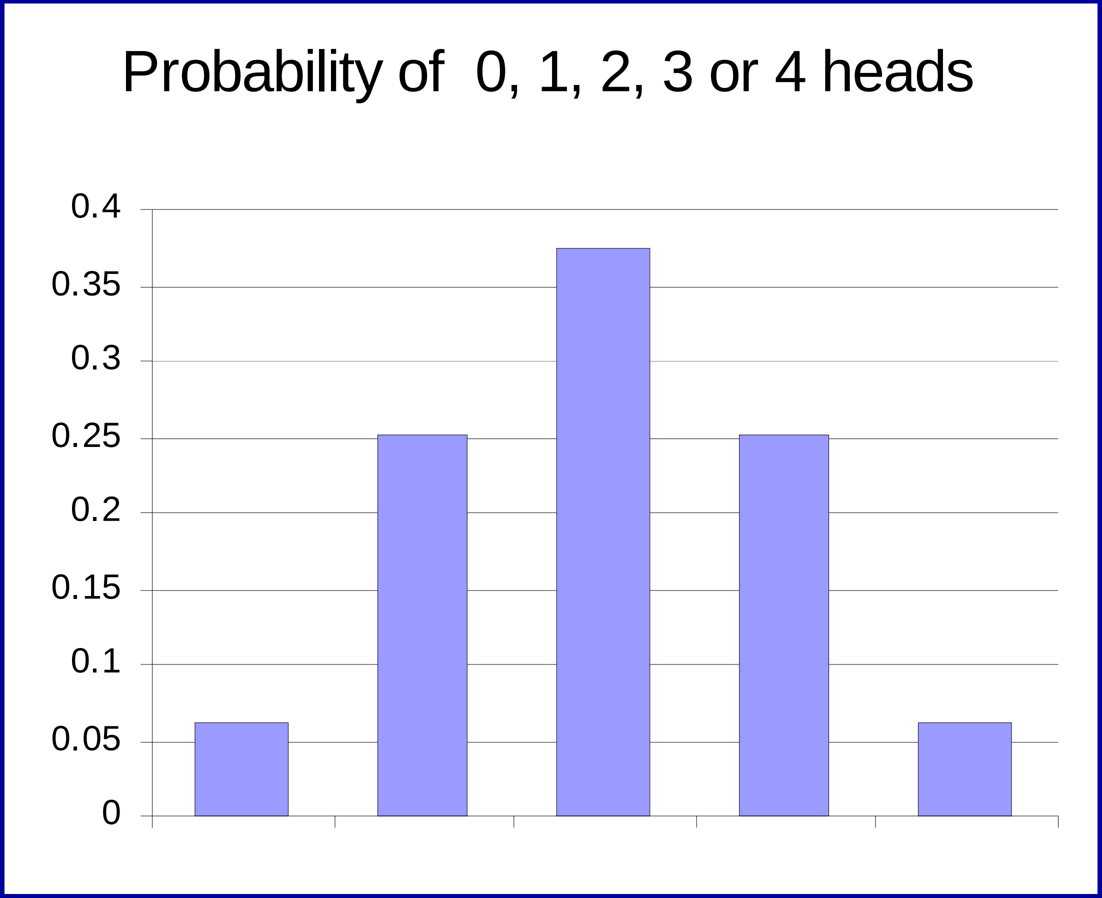
Outcome	#Heads	Outcome	#Heads
TTTT	0	THTH	2
TTTH	1	HTTH	2
TTHT	1	HTHT	2
THTT	1	HHHT	3
HTTT	1	HHTH	3
TTHH	2	HTHH	3
THHT	2	THHH	3
HHTT	2	HHHH	4

Probabilities

- No heads : $1 \times (1/2^4)$ = 1/16
- 1 head : $4 \times (1/2^4)$ = 4/16 = ¼
- 2 heads : $6 \times (1/2^4)$ = 6/16 = 3/8
- 3 heads : $4 \times (1/2^4)$ = 4/16 = ¼
- 4 heads : $1 \times (1/2^4)$ = 1/16

- Total : 16/16 = 1.0

Probability distribution



Coin tossed *n* times

- What is probability to have
  - No heads
  - 1 head
  - 2 heads
  - 3 heads
  - ...
  - $(n - 1)$ heads
  - n heads
- If the number of outcomes leading to **k** heads is known, the answer is straightforward

Number of outcomes with k heads

- Number of outcomes leading to k "heads" out of n trials is given by binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Probability to have k heads

- Every outcome has probability $1/2^n$

- The probability to have *k* "heads" is
  - $(1/2^n)$ x [# of outcomes leading to *k* "heads"]

- P(**k** heads in **n** trials) =

$$\binom{n}{k} \cdot \left(\frac{1}{2}\right)^n = \frac{n!}{k! \cdot (n-k)!} \cdot \left(\frac{1}{2}\right)^n$$

Problem

- Compute binomial probabilities for $n = 12$ and
  - $k = 5$
  - $k = 9$
  - $k = 10$

Solution

- $k = 5$

  12! / (5! 7!) = 8x9x10x11x12 / 2x3x4x5 = 792
- $k = 9$

  12! / (9! 3!) = 10x11x12 / 2x3 = 220
- $k = 10$

  12! / (10! 2!) = 11x12 / 2 = 66

Problem – Fair game?

- We toss a coin 12 times. If there are 10 or more heads, **YOU** get **€100** (!!!), otherwise you pay me only €5

- Question – are going to bet with me (your aim is to make some money)?

Solution

- What is probability to have 10 or more heads in 12 coin-tosses?
- $P(k \geq 10) = P(k=10) + P(k=11) + P(k=12) = 66 / 4096 + 12 / 4096 + 1 / 4096 = 79 / 4096 \approx 1.9\%$
- Bet of 20 : 1 does not look fair as chances are 50:1!

- In 100 games, I will loose ~2 times (debit € 200) and win ~98 times (credit € 480). Good profit!

Problem

- Probabilities of observing a person with genotypes NN, DN and DD in a population are
  - P(NN)=0.81, P(ND)=0.18 and P(DD)=0.01

- What number of carriers are expected in this sample?
- What is the probability that in a sample of 10 random people there will be
  - Exactly one carrier
  - One or more carrier(s)

Expected number of carriers

- P(carrier) = P(**ND**) + P(**DD**) = **p** = 0.19
- Expected number is 0.19 x 10 = 1.9 people

P(exactly one carrier)

- P(carrier) = P(**ND**) + P(**DD**) = **p** = 0.19
- It can be that we sample a carrier first and then sample 9 non-carriers. As the events are independent, probability to have such a sample is 0.19×0.81^9
- There are 10 outcomes leading to 1 carrier
- Therefore probability to have exactly one carrier is

  $0.19 \times 0.81^9 \times 10 = 0.29$

P(at least one carrier)

$P(\geq 1 \text{ carrier}) = Pr(1 \text{ or } 2 \text{ or } 3 \text{ or } \ldots 10) =$

$1 - Pr(0 \text{ carriers}) =$

$1 - 0.81^{10} = 0.88$

Binomial distribution

- Heads and tails:
  - P(success) = **p**
  - Pr(failure) = 1 – **p**
- Probability of an outcome with **k** successes out of **n** trials
  - **k** successes => (**n** – **k**) failures
  - Events are independent, thus probability of an outcome with **k** successes is $p^k (1-p)^{(n-k)}$
  - The number of outcomes with **k** successes are given by binomial coefficients

Binomial distribution

- The probability to have *k* successes if probability of a success is *p* and number of trials is *n* is given by

$$P_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- with mean equal to *n p*
- and variance = *n p (1 – p)*

Formal answers

- Parameters: $p = 0.19$, $n = 10$
- Exactly one carrier (k=1)
  - $10!/(9!1!)\ 0.19^{1}\ 0.81^{10-1} = 0.29$
- At least one carrier (k>0)
  - 1 – P(0 carriers) =

$$1 - 10!/(10!0!)\ 0.19^{0}\ 0.81^{10} =$$

$$1 - 0.81^{10} = 0.88$$