Binomial distribution

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GE02: day 1 part 2

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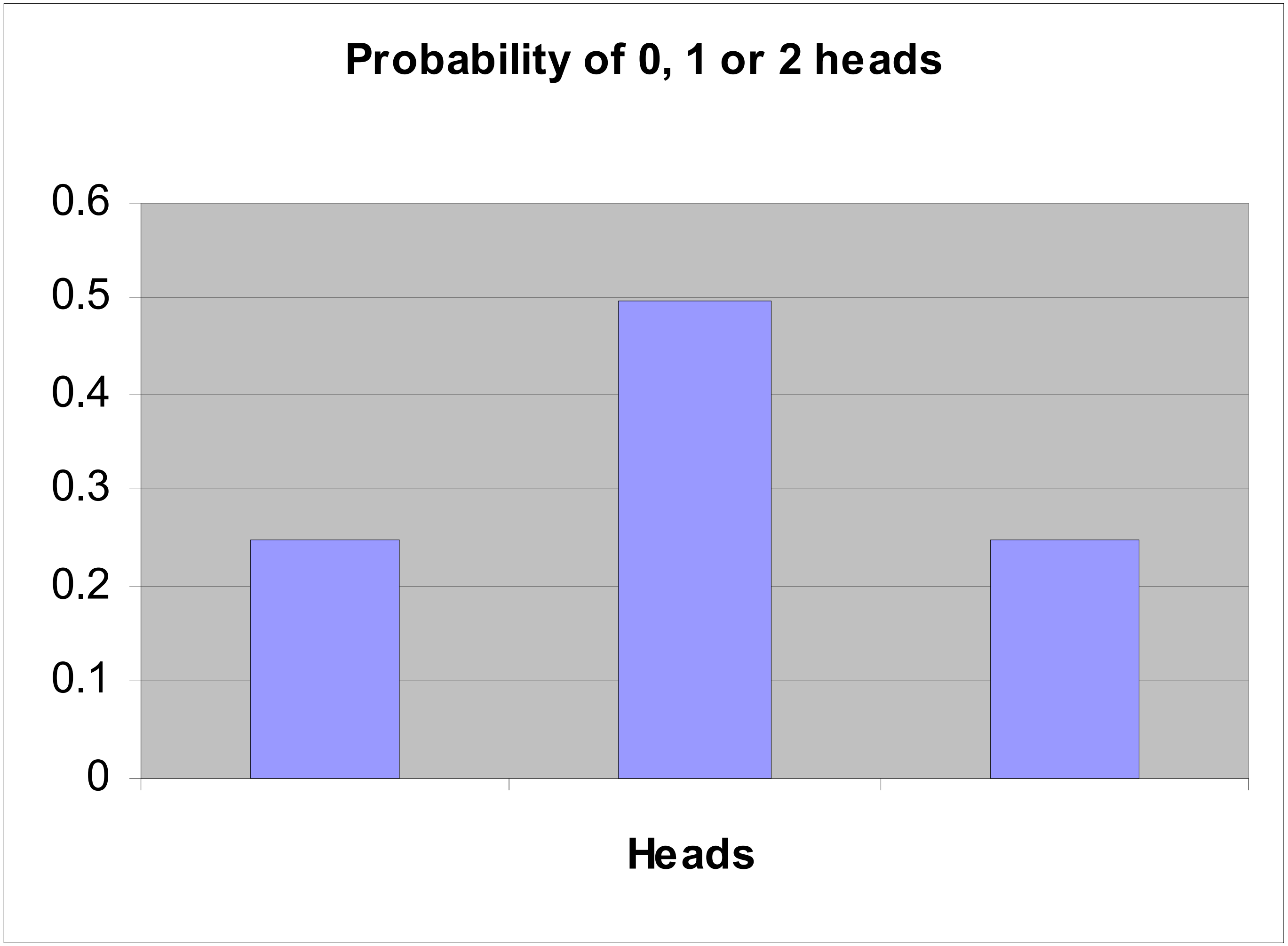
Problem

- Consider an experiment in which a coin is tossed 2 times. What is probability to have
  - No heads
  - 1 head
  - 2 heads

Solution

- Probability of any configuration is
  - 2^{-n} = ½ ½ … ½ (*n* times)
  - Four configurations possible – HH, HT, TH and TT, each having probability ¼
- No heads ¼
- 1 head ¼ + ¼ = 2 ¼ = ½
- 2 heads ¼
- Total 4 ¼ = 1.0

Probability distribution



Task

- Consider an experiment in which a coin is tossed 4 times. What is probability to have
  - No heads
  - 1 head
  - 2 heads
  - 3 heads
  - 4 heads

Idea of solution

- Every configuration has probability $(½)^4 = 1/16$

- The probability to have *k* "heads" is
  - $(½)^4$ x [# of configurations leading to *k* "heads"]

- Need to write down all 16 configurations

16 configurations

Config	#Heads
TTTT	0
TTTH	1
TTHT	1
THTT	1
HTTT	1
TTHH	2
THHT	2
HHTT	2

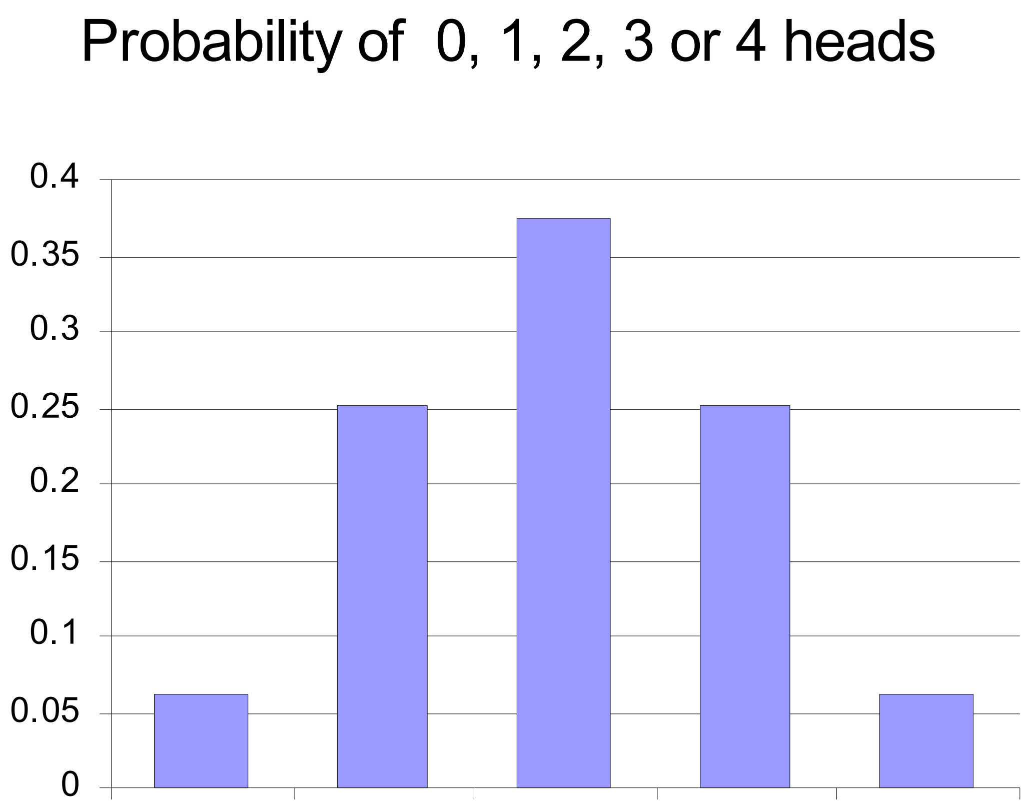
Config	#Heads
HHHH	4
HHHT	3
HHTH	3
HTHH	3
THHH	3
HTHT	2
THTH	2
HTTH	2

Probabilities

- No heads : $1 \times (1/2^5) = 1/16$
- 1 head : $5 \times (1/2^5) = 4/16 = ¼$
- 2 heads : $10 \times (1/2^5) = 6/16 = 3/8$
- 3 heads : $10 \times (1/2^5) = 4/16 = ¼$
- 4 heads : $5 \times (1/2^5) = 1/16$

- Total : 16/16 = 1.0

Probability distribution



Coin tossed n times

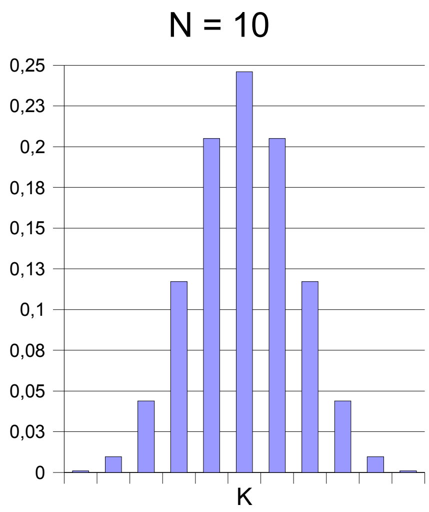
- What is probability to have
  - No heads
  - 1 head
  - 2 heads
  - 3 heads
  - ...
  - $(n - 1)$ heads
  - n heads

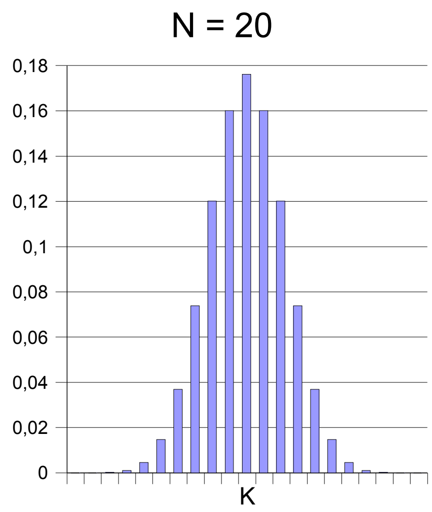
Solution: binomial coefficients

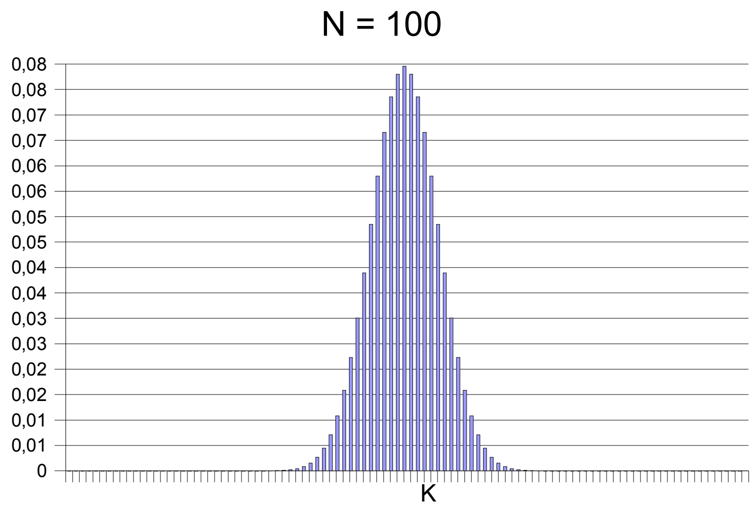
- Every configuration has probability $1/2^n$
- The probability to have *k* “heads” is
  - $(1/2^n)$ x [# of configurations leading to *k* “heads”]
- Number of configurations leading to *k* “heads” out of *n* trials is given by binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Binomial distribution at different *n*







Task

- Compute binomial coefficient for $n = 12$ and
  - $k = 5$
  - $k = 9$
  - $k = 10$

Answer

- $k = 5$: 12! / (5! 7!) = 8 9 10 11 12 / 2 3 4 5 = 792
- $k = 9$: 12! / (9! 3!) = 10 11 12 / 2 3 = 220
- $k = 10$: 12! / (10! 2!) = 11 12 / 2 = 66

Task – Fair game?

- We toss coins 12 time. If there are 10 or more coins, you pay me €1, otherwise I pay you 1 cent.
- Question – is that a fair game?

Answer

- What is probability to have 10 or more heads in 12 coin-tosses?
- $P(k>9) = P(k=10) + P(k=11) + P(k=12) = 66 / 4096 + 12 / 4096 + 1 / 4096 = 79 / 4096 \approx 1.9\%$
- Bet of 100 : 1 does not look fair as chances are 50:1!

Problem

- Probabilities of observing a person with genotypes NN, DN and DD in a population are
  - P(NN)=0.81, P(ND)=0.18 and P(DD)=0.01
- What is the probability that in a sample of 10 random people there will be
  - Exactly one carrier
  - One or more carrier
  - What number of carriers are expected in this sample?

Probability to have exactly one carrier

- P(carrier) = P(ND) + P(DD) = p = 0.19
- It can be that we sample a carrier first and then sample 9 non-carriers. As the events are independent, probbaility to have such a sample is 0.19×0.81^9
- There are 10 configurations leading to 1 carrier
- Therefore probability to have exactly one carrier is

$$0.19 \times 0.81^9 \times 10 = 0.29$$

Probability to have at least one carrier

$P(\geq 1 \text{ carrier}) = Pr(1 \text{ or } 2 \text{ or } 3 \text{ or } \ldots 10) =$

$1 - Pr(0 \text{ carriers}) =$

$1 - 0.81^{10} = 0.88$

Binomial distribution

- Heads and tails:
  - P(success) = p
  - Pr(failure) = $1 - p$
- Probability of a configuration with K successes out of N trials
  - k successes => $(n - k)$ failures
  - Events are independent, thus probability of a configuration with k successes is $p^k (1-p)^{(n-k)}$
  - The number of configurations with k successes are given by binomial coefficients

Binomial distribution

- The probability to have k successes if probability of a success is p and number of trials is n is given by

$$P_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

- with mean equal to $n\,p$
- and variance = $n\,p\,(1 - p)$

Formal answers

- Parameters: $p = 0.19$, $n = 10$
- Exactly one carrier
  - $10!/(9!1!)\ 0.19\ 0.81^9 = 0.29$
- At least one carrier
  - $1 - P(0 \text{ carriers}) = 1 - 10!/(10!0!)\ 0.81^{10} = 0.88$
- Expected number of carriers is
  - $n\,p = 0.19 \times 10 = 1.9$

Task

- You cannot see 1.9 carriers
- What is more frequent, 2 or 1?

Two carriers appear more frequent